Greg Kuperberg's Lectures on

Introduction to Quantum Information Theory

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Contents

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1.1 Modelling of TPCPs

Let's consider two finite-dimensional von Neumann algebras $\mathcal{A}$ and $\mathcal{B}$. Our goal is to model $\mathrm{TPCP}(\mathcal{A}, \mathcal{B})$. For such TPCPs E,

$$E \colon \mathcal{A}^{\#} \to \mathcal{B}^{\#} \text{ s.t. } E(\rho)(1) = \rho(1)\ \forall \rho \in \alpha^{\#}.$$

The set of quantum operators $\mathrm{TPCP}(\mathcal{A}, \mathcal{B})$ which acts on elements in $\mathcal{A}$ and produces elements in $\mathcal{B}$ is a *covex body* and its codimension is $\dim \mathcal{A}$. In particular, the codimension is not 1 in general and the set isn't the *base* of $\mathrm{CP}(\mathcal{A}, \mathcal{B})$. (Ref: arXiv:0710.1571)

The geometry of the convex body $\mathrm{TPCP}(\mathcal{A}, \mathcal{B})$ is complicated even in the dimension $d = 2$ case of a qubit. It's more common to study the much simpler cone $\mathrm{CP}(\mathcal{A}, \mathcal{B})$. Some form of Stinespring's theorem holds here too and the TP part works out automatically.

Furthermore, $\mathrm{CP}(\mathcal{A}, \mathcal{B})$ is the same cone, non-uniquely, as $(\mathcal{A}\overline{\otimes}\mathcal{B})^{+}$. This denotes the cone of positive (normalizable though not necessarily normalized) states on the completed tensor product $\mathcal{A}\overline{\otimes}\mathcal{B}$. Thus, this cone lies in the predual $(\mathcal{A} \otimes \mathcal{B})^{\#}$.

For example, let's consider an $E \in \mathrm{CP}(a\mathbb{C}, b\mathbb{C})$ (note: $\mathrm{CP} = \mathrm{P}$, classically). In the classical case, E is a $b \times a$ matrix with all positive entries and it can be identified with states in $(ab\mathbb{C})^{+} \cong a\mathbb{C} \otimes b\mathbb{C}$.

1.2 Kraus Decomposition

Let $\mathcal{A}$ be $M(a)$ and let $\mathcal{B}$ be $M(b)$. Let E be CP s.t. $E \colon \mathcal{A}^{\#} \to \mathcal{B}^{\#}$. Then the action of the quantum operation E on the state ρ can be expressed as:

$$E(\rho) = \sum_j x_j \rho x_j^*$$

for some set of operators $\{x_j\}_j$ satisfying $\sum_j x_j^* x_j = \mathbf{1}$ (TP) where $\mathbf{1}$ is the identity operator.

For e.g., if $\mathcal{A} = \mathcal{B} = M(d)$ then $E(\rho) = u\rho u^*$ is unitary (actually, TPCP).

$$\mathrm{CP}(M(a)^{\#}, M(b)^{\#})) \cong (M(a) \otimes M(b))^{+}$$

The extremal ray maps in $\mathrm{CP}(M(a)^{\#}, M(b)^{\#}))$ are of the form $x\rho x^*$ and they correspond to the extremal rays ("pure states") $|\psi\rangle\langle\psi|$ in $(M(a) \otimes M(b))^{+}$. (Ref: Quantum Theory from First Principles)

1.3 Choi-Jamiolkowski Isomorphism

This involves converting a channel from $\mathcal{A}$ to $\mathcal{B}$ as a state on $\mathcal{A} \otimes \mathcal{B}$ without loss of parameters, with the following interpretation $E \colon \mathcal{A}^{\#} \to \mathcal{B}^{\#}$. $\rho_2 \in (\mathcal{A} \otimes \mathcal{A})^{+}$ is a copied state on $\mathcal{A}$ with full support. Define

$$\rho_{\text{Choi}} \colon = (E \otimes \mathrm{Id})(\rho_2).$$

Theorem (Choi): Given ρ_2, $E \mapsto \rho_{\text{Choi}}$ is an isomorphism of cones.

1.4 C* Algebra vs. VNA Approach

The main differences between the VNA and C* Algebra approach has been summarized below.

	Observables	States
C* Algebra	$\mathcal{A}$ few	$\mathcal{A}^*$ many
von Neumann Algebra	$\mathcal{A}$ many	$\mathcal{A}^\#$ few

In the $[0, 1]$ case,

	Observables	States
C* Algebra	$C([0,1])$	$C([0,1])^*$
von Neumann Algebra	$L^\infty([0,1])$	$L^1([0,1])$

1.5 Exercises

Consider $\mathrm{TPCP}(M(2), M(2))$. Rescale $M(2)^\Delta$ concentrically by $s \in \mathbb{R}$, fixing

$$\rho = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix}.$$

This operation is TP for free. When is it TPCP?

1) TPP when $-1 \leq s \leq 1$.
2) $s = -1$ is our counterexample.
3) $s \geq 0$ is depolarization, is TPCP.
4) $-\frac{1}{3} \leq s \leq 1$ is TPCP.