Greg Kuperberg's Lectures on

Quantum Error Correction

Sanchayan Dutta
dutta@ucdavis.edu

1 Lecture X (21 April, 2022)

1.1 Preliminaries

Let A be a finite alphabet. Say $E\colon l^1(A) \to l^1(A)$ be a stochastic matrix. If A is a set, then $l^1(A)$ represents functions from $A \to \mathbb{C}$ such that $\sum_i |f(a_i)| < \infty$ where $a_i \in A$. The key point is that $l^1(A)^* \cong l^\infty(A)$. We know $||f||_1 = \sum_{x\in A} |f(x)|$ and $||g||_\infty = \sup_{x\in A}|g(x)|$. There is an immediate argument that $l^\infty(A)$ lies in the dual of $l^1(A)$. It's harder to show that it's all of it. For one, if you take the dot product of a convergent sequence and a bounded sequence you get a convergent dot product. $l^\infty(A)$ means bounded sequences on the set A.

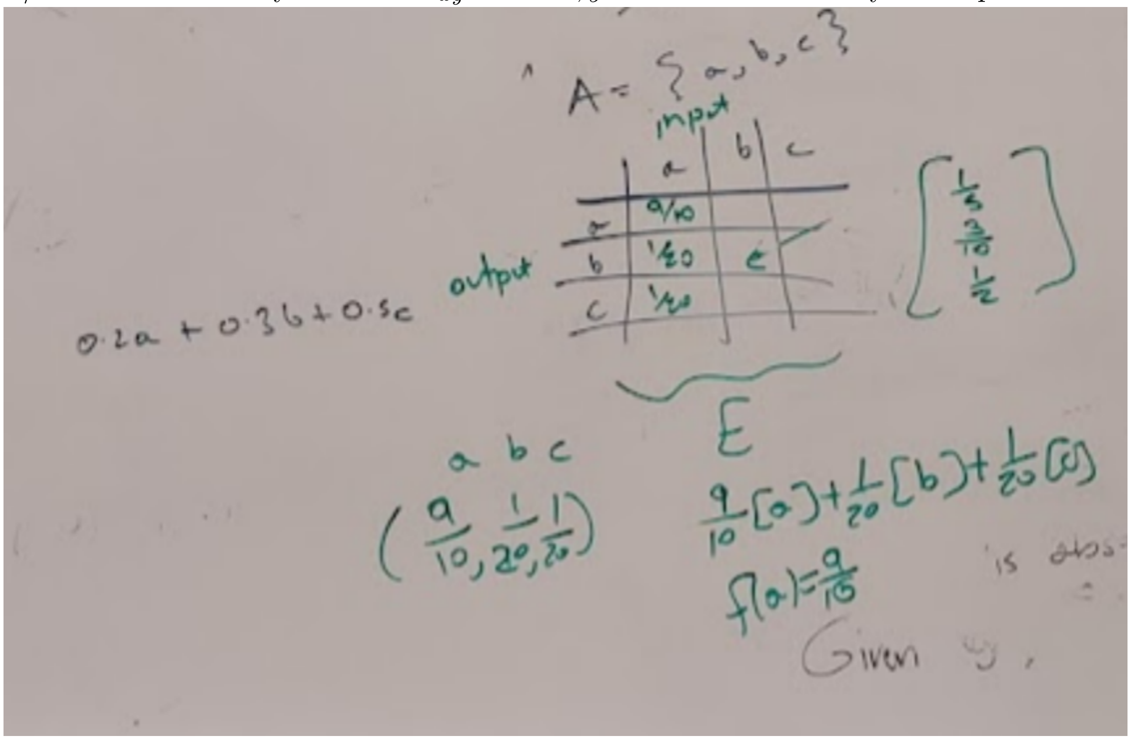
To show $l^1(A) \subseteq l^\infty(A)$ recall that if $||f||_1 < \infty$ and $||g||_\infty < \infty$ then $\sum_{x\in A} |f(x)g(x)| < \infty \implies \langle g|f\rangle < \infty$ is absolutely convergent (A is not ordered so we need absolute convergence). Given $g \in l^\infty$ we get a linear map $\langle g|$ from $l^\infty(A)^* \to l^1(A)$ and we want to show that it is both injective and surjective. Injectivity is easier – use a delta function argument. Surjectivity is harder – if you have a continuous bra then it comes from an l^∞ function. You can use the axiom of choice to make a discontinuous bra which is not of the $l^\infty(A)$ form. The fact that it is surjective is an example of the Riesz representation theorem. Every continuous dual vector has some specific form was noticed by Riesz. There's this Banach space fact – whether it's functionals or linear operators – continuous is equivalent to Lipchitz.

The word among functional analysts for bounded is Lipchitz. You get a certain triangle inequality $|\langle g|f\rangle| \leq ||f||_1||g||_\infty$ which says that g is a bounded functional or bounded bra. The converse couldn't have worked if you had a non-bounded functional. Now bounded functionals are continuous but there is this cosmically general converse due to Banach that all continuous functionals are bounded operators. This was in Banach's Ph.D. thesis. Though Riesz was mainly concerned with the l^2 case where $l^2(A)^* \cong l^2(A)$. One important point is that $l^\infty(A)^* \supset l^1(A)$ strictly and so it's not exactly Riesz type due to lack of reflexivity. The dual of $l^\infty(A)$ is much larger than its pre-dual $l^1(A)$. (Note: $\mathcal{L}(H)$ has an operator norm and that norm is also used for a VNA subalgebra.)

Let X and Y be two Banach spaces. Then $\mathcal{L}(X, Y)$ has an operator norm. The norm of A the $\sup_{||x||=1} ||Ax||$. This operator norm is not called an ∞-norm. The reason being that in full generality there is no notion of 2-norm or 1-norm. There is no parameter and it's just this norm which is defined by hypothesis. $\mathcal{L}(X, Y)$ is a Banach space where the operator norm is bounded. X^* is by definition $\mathcal{L}(X, \mathbb{C})$.

1.2 Stochastic Matrices

Let A be an alphabet, say finite for now. Say $E\colon l^1(A) \to l^1(A)$ is stochastic. E is a matrix with rows/columns labelled by A. Then E_{xy} where $x, y \in A$ is a matrix entry. Example:



Note: $\mathcal{L}(X, Y)$ is a certain completion of $X^* \otimes Y$ in a very general sense.

$l^1(A) = l^\infty(A)^\# \supseteq l^\infty(A)^\Delta$ and $E(l^\infty(A)^\Delta) \subseteq l^\infty(A)^\Delta$. In general, $l^1(A)$ and $l^\infty(A)$ are defined over the complex numbers $\mathbb{C}$. The state region $l^\infty(A)^\Delta$ spans the real part of $l^\infty(A)$. (The stochastic matrices consist of probabilities so they must contain real numbers in $[0, 1]$.) $l^\infty(A)$ is a commutative algebra because you can only multiply pointwise. The elements of $l^\infty(A)$ are like density matrices. This a baby version of the fact that the diagonal entries of a density matrices are real. In the commutative case, you can think of all diagonal density matrices. The booleans in $l^\infty(A)$ are exactly the idempotents – when the values are 0s and 1s. This is a reminder that we're doing probability theory. We can throw in quantum states in this way: $l^\infty(A) \subseteq L(H)$.

$l^\infty(A)$ consists of the observables (we're drawing a parallel to the harder quantum case). To clarify our notation, if $\mathcal{M}$ is the algebra of observables and $\mathcal{M}^\#$ is the predual of $\mathcal{M}$ then $\Pr[b]\colon = \rho(b)$. Then $\rho \geq 0$ means $\rho(b) \geq 0 \forall b$. If $\mathcal{A}$ and $\mathcal{B}$ are commutative then $E\colon \mathcal{A}^\# \to \mathcal{B}^\#$ is stochastic: meaning E is TPP. E is positive means $E(\mathcal{A}^+) \subseteq \mathcal{B}^+$. TP in addition means $E(\mathcal{A}^\Delta) \subseteq \mathcal{B}^\Delta$.

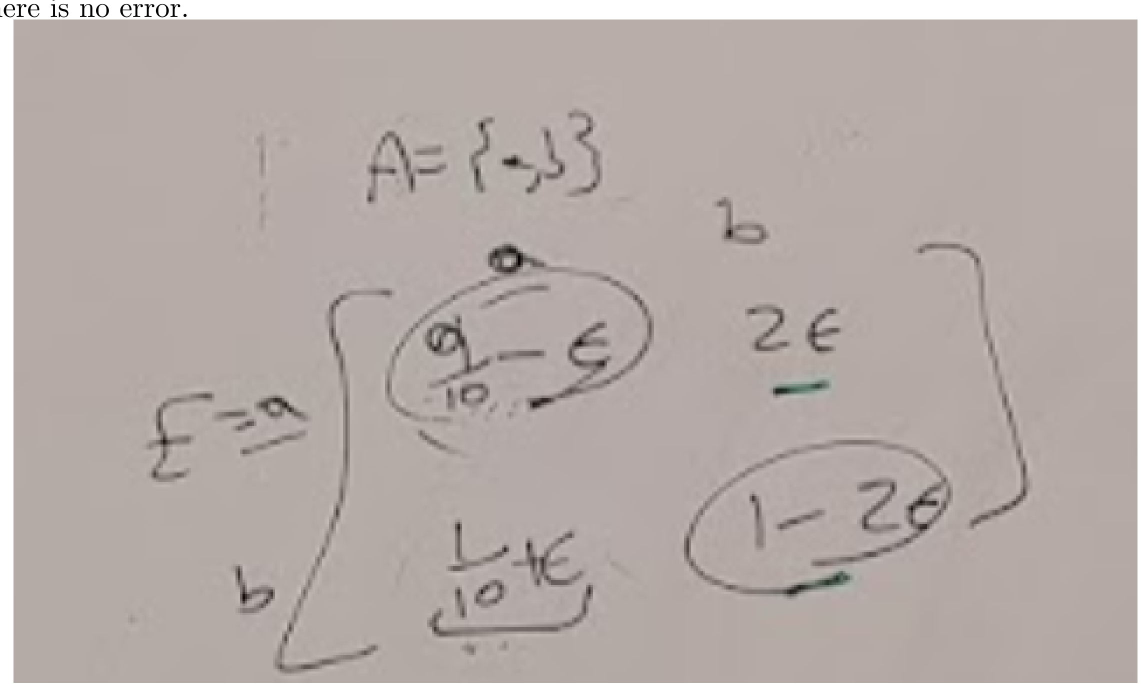
If $\mathcal{A}$ and $\mathcal{B}$ are commutative, then $E\colon \mathcal{A}^\# \to \mathcal{B}^\#$ means E is stochastic means that E is TPP. By definition, all predual ρ, s.t. the $\mathrm{Tr}(\rho) = 1$ and $\rho \geq 0$. So E is positive means that $E(\mathcal{A}^+) \subseteq \mathcal{B}^+$ and further E is TP means $E(\mathcal{A}^\Delta) \subseteq \mathcal{B}^\Delta$. The booleans b lie in $\mathcal{M}_{\mathbb{Z}/2}$.

1.3 Evaluation Ring $\mathbb{R}[[\epsilon]]$

Let $\mathbb{R}[[\epsilon]]$ be a ring of formal power series in ϵ. It is an ordered ring $0_{\mathbb{R}[[\epsilon]]} < \epsilon < a \in \mathbb{R}_{>0}$. We know $\sum_{n=0}^\infty a_n \epsilon^n > 0_{\mathbb{R}[[\epsilon]]}$ iff the 1st non-zero coefficient $a_n \in \mathbb{R}$ is $> 0_\mathbb{R}$. For example, $\epsilon^2 - 1000\epsilon^3 > 0_{\mathbb{R}[[\epsilon]]}$. Since it's a ring embedding $0_\mathbb{R}$ is sent to $0_\mathbb{R}[[\epsilon]]$.

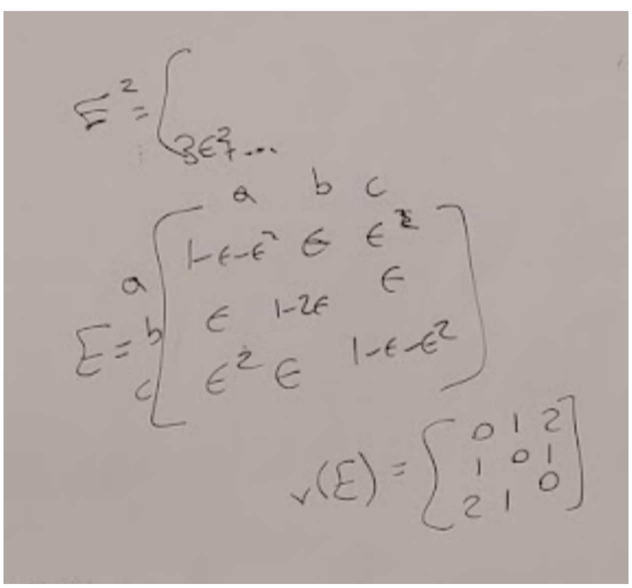
A tropicalized stochastic matrix $E\colon l^1(A)[[\epsilon]] \to l^1(A)[[\epsilon]]$ has entries $E_{xy} \in \mathbb{R}[[\epsilon]]$. We call E

stochastic if columns sums are 1 and entries are ≥ 0. This can be many things. If in addition $E \equiv I \bmod \epsilon$ then we can interpret E as a model of rare error. This is saying that we set ϵ to 0 then there is no error.



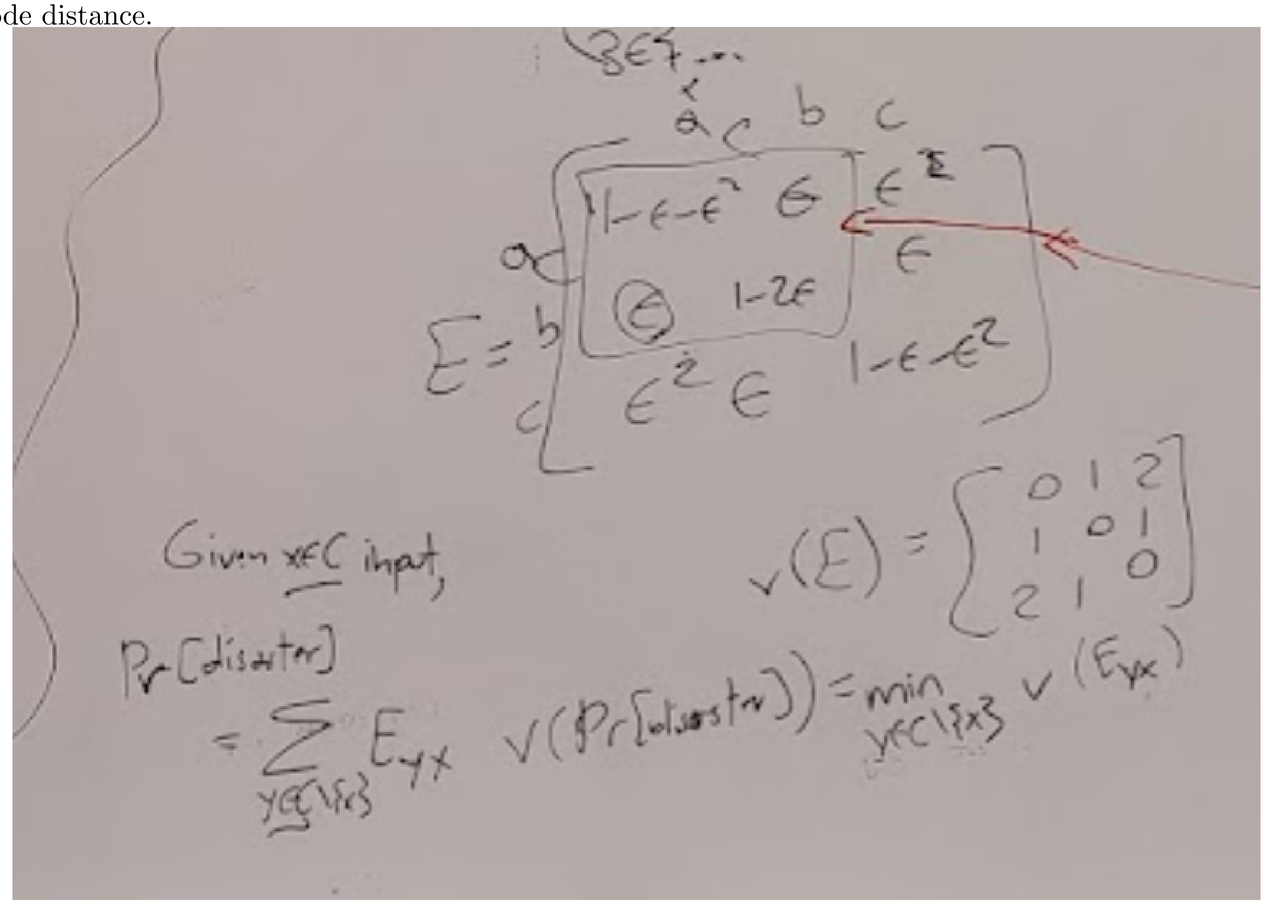
Call E stochastic if the column sums are 1 and entries are ≥ 0. This can be many things. If in addition $E = I \bmod \epsilon$, then we can interpret E as a model of rare error. Any $p \in \mathbb{R}[[\epsilon]]$ has a valuation $v(p) \colon =$ degree of 1st non-zero term $=$ # of times that ϵ divides p. Any E has a valuation matrix $v(E)$, where you just take the valuation of each entry. We can sometimes conclude that $d(x, y) \colon = v(E)_{xy} = v(E_{xy})$ is a metric on A.

1. E models rare errors, i.e., $E \equiv I \bmod \epsilon \iff d(x, y) = 0$ iff $x = y$.
2. E has approximate time-reversal symmetry. E and E^T are on the same scale, i,e., $v(E) = v(E^T) \iff d(x, y) = d(y, x)$.
3. E and E^2 cause error at the same scale $\iff d(x, z) \leq d(x, y) + d(y, z)$.



Higher valuation numbers means the events are more unlikely – freak events. Say, a valuation of 2 means the event is as probable as 2 coins standing on an edge upon being tossed simultaneously.

We can rate a code $C \subseteq A$ by the valuation $v(\Pr(\text{Undetected error})) \overset{\text{thm}}{=} \min_{x,y\in C, x\neq y} d(x,y) \overset{\text{def}}{=}$ code distance.



If $p, q \in \mathbb{R}[[\epsilon]]$ then $v(p+q) = \min(v(p), v(q))$. Transitions from code to code itself is an irrecoverable error. Otherwise it is called a recoverable error.

Examples: The Hamming metric counts how many positions are different. The Lee metric counts how many positions differ as well as how much each position differs. Usual Hamming metric is defined on $\{0,1\}^n$ and the general Hamming metric is on A^n for all A. The Lee metric is defined on $(\mathbb{Z}/k\mathbb{Z})^n$ generated by changing one letter ± 1. Manhattan is like Lee metric but on $\mathbb{Z}^n$.

Facts:

1. You can tropicalize TPCPs too (and CPs).
2. Some axioms for E yeild KW style quantum metric spaces. $\rho \mapsto V\rho V^*$.

Say $V = \epsilon^3 X + \epsilon^6 Y$; these are ϵ-perturbations in the Kraus terms. (There is a way define valuation of a Kraus terms and CPs; not just TPCPs.) Suppose you know how to fix an X error and you have a Kraus term like $\rho \mapsto V\rho V^*$ and you can fix X in $\epsilon^3 X$ but $\epsilon^6 Y$ then you can happy. There can be quantum superposition of $\epsilon^3 X$ and $\epsilon^6 Y$. If you can correct X then you can *mostly* correct the Kraus term $\epsilon^3 X + \epsilon^6 Y$ because it's pretty close to X. The distance between this Kraus terma and X is small. What's left is even smaller, i.e., $\epsilon^6 Y$ because the valuation is much smaller. There are two types of errors here – the ones that you can fix like $\epsilon^3 X$ and the ones that you don't really care about like $\epsilon^6 Y$ because they have a much larger valuation.

Errors like these can be both in classical or quantum superposition – both of those things can happen. And you also have two kinds of errors – the ones you can fix and the ones that have too high of a valuation for us to care about.

There is a trichotomy – non-error, recoverable error and irrecoverable error. In the classical case, these can occur only in classical superposition whereas in the quantum case these can occur both in classical and quantum superposition. (Non-recoverable errors are error transitions from C to C – these can also be called disasters.) The way that you handle non-recoverable error is that you just assume that its valuation is high enough that you don't have to worry about it. Whether an error is recoverable or non-recoverable depends on how you have designed the code, say like the parity check.

Quantum error correction gives you a gift – the error correction techniques will work whether they occur in classical or quantum superposition once you can say correct X, Y and Z errors (for example). You can interpret ϵX as a rare X error. There is a way to do this.

Imagine an erroneous process that is a single Kraus term because it is a unitary operator. Then the Bloch sphere rotates slightly about the x-axis. Intuitively, at first, that's an error that definitely took place. But that is also a quantum superposition of no rotation and a rotation of the Bloch sphere by 90 degrees.

Example of a tropicalized unitary error: